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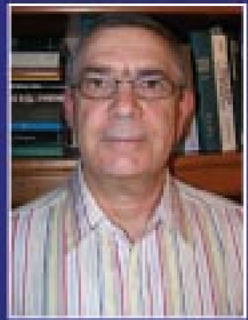

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$$A^{(1)}_{sm}T^{(12)}_{mn} = A^{(2)}_{sn}T^{(21)}_{nm}$$

$$A^{(1)}_{sm}R^{(1)}_{mn} = A^{(1)}_{sn}R^{(1)}_{nm}$$

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Abstract

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Chapter 1.
Introduction

We will examine basic discontinuities in cylindrical acoustic wave-guide with rigid wall and at condition of axial symmetry. A medium inside this wave-guide is a liquid or a gas. Basic discontinuities in such wave-guide are 'diameter step' and 'rigid diaphragm with coaxial opening' – see Figure 1. These discontinuities were considered recently [1] for case of single traveling mode (and at condition of axial symmetry). Now we will follow further with numerical analysis of the reciprocity properties of these discontinuities in multi-mode case. But numerical analysis of the reciprocity properties may be possible with accurate numerical data for transmission/reflection coefficients. Hence, our first goal here should be numerical analysis of transmission/reflection coefficients for above basic discontinuities. After it we can proceed – and it will be done below – with numerical analysis of the reciprocity properties of the same discontinuities.

Let us consider first discontinuity – diameter step. It needs to build computer program for numerical analysis of wave fields in all parts of this structure. There are following waves in the structure: incident wave goes from the larger tube to the junction with the smaller tube and generates reflected waves in the larger tube and transmitted wave(s) in the smaller tube. Analogous situation should be if the incident wave interacts with the discontinuity from the smaller tube. But real numerical analysis here splits on two different ways: transmission 'into smaller tube' and transmission 'into larger tube'. Some details in the procedure of numerical analysis over these two ways are different (it needs to build two different computer programs for numerical analysis of these two versions of the structure 'diameter step'). Hence, these two ways will be considered separately as numerical problems for 2 different structures – as it is shown on Figure 1, A and B, correspondently. Mathematical differences between these two structures are generated by the boundary conditions at the discontinuity – see sections 2 and 3 for details. We will follow to one of most traditional way in the acoustics: we will build solution in each section of the structure and we will glue them at the discontinuity. Total numerical solution over this way transforms to solution of a system of linear equations. The boundary conditions at the discontinuity define structure of such system.

The boundary conditions at the discontinuity 'rigid diaphragm with coaxial opening' – see Figure 1, C – are same at alternation of incident wave direction. Hence, the numerical analysis of such discontinuity may be done with single computer program – see chapter 4 below. So, numerical analysis of two basic discontinuities in the acoustic wave-guide leads to 3 different mathematical problems (or: it leads to 3 different structures that should be analyzed separately).

Each structure contains two sections: section 1 – it is a left half of the structure and section 2 – it is its right half. A discontinuity is a junction of these two sections. Incident waves are generated in the section 1 and they travel from left to right – always. Origin of the coordinate system (z, r) is at the junction, z-axis goes from left to right along the axis of symmetry of the wave-guide (the incident wave goes in positive direction of this axis).

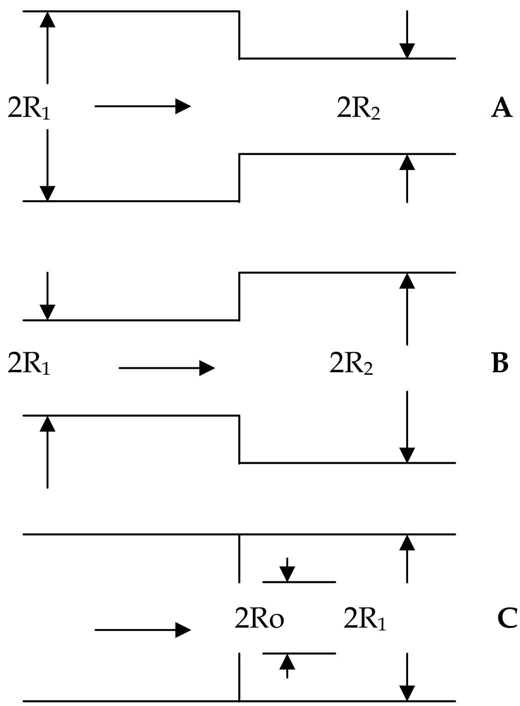

Figure 1. Basic types of discontinuity in cylindrical wave-guide. A – dia-meter step, transmission into smaller tube; B – diameter step, transmission into larger tube; C – rigid diaphragm with coaxial opening.

Rigorous descriptions of all possible traveling modes in cylindrical wave-guide we can find, for example, in books of Morse & Ingard [2] and Skudrzyk [3] – last book contains more details. These books provide accurate expressions for wave fields in uniform cylindrical wave-guide (without any discontinuity). First approximate solutions (1-D model) for discontinuity 'diameter step' were obtained and presented by Miles [4] and Karal [5]. Now their simple formulas can be found in almost any book about acoustics

$$R_0 = \frac{S_1 - S_2}{S_1 + S_2} \qquad T_0 = \frac{2 S_1}{S_1 + S_2} \tag{1}$$

where: Ro and To are reflection and transmission coefficients, correspondently; S_1 and S_2 are areas of cross section of the wave-guide – in the sections 1 and 2 for the discontinuities on Figure 1. These well known approximate formulas had been built for the low frequency case. So, the formulas (1) should be accurate if work frequency (f) is much less than frequency of the first radial resonance (f_{r1}) in the both sections of the structure.

$$f << f_{r1} \qquad f_{r1} = \frac{\gamma_1 C}{2 \pi R} \tag{2}$$

where: C is wave velocity in an acoustic medium inside the wave-guide. More accurately: C is a velocity of the plane compression waves in unbounded acoustic medium, identical with the medium inside the wave-guide. Coefficient γ_1 is a first not zero root of equation

$$\frac{\partial}{\partial r} J_0(\alpha_m r) = -\alpha_m J_1(\alpha_m r) = 0 \quad at \;\; r = R \tag{3}$$

Above equation follows from the boundary condition at a rigid wall in the cylindrical wave guide (at r = R) for radial component of medium velocity (should be zero at a rigid wall) [2, 3]. The roots γ_m provide radial wave numbers α_m (with same index m)

$$\begin{aligned} &\alpha_m R = \gamma_m \; ; \quad m = 0, 1, 2, \ldots \quad \rightarrow \quad \alpha_m = \gamma_m / R \\ &\gamma_0 = 0, \;\; \gamma_1 = 3.832, \;\; \gamma_2 = 7.016, \ldots \end{aligned} \tag{4}$$

Recent work [1] provides data for real accuracy of expressions (1): errors do not exceed 5% (for the modules of R_0 and T_0) in the frequency range

$$0 < f < (0.2 - 0.5) f_{r1}. \qquad (5)$$

Present work employs the same technique, except technique to solve a system of linear equations. All systems of linear equations are solved here in very traditional way. Not traditional technique in work [1] provided, probably, some difficulties for readers.

So, let us consider wave field for the velocity potential in the section 1 (z < 0) with radius R_1. Time factor is $exp(+i\omega t)$ – all expressions here are built in the system with positive time. It is assumed that there can exist two traveling modes in the section 1 – with indexes m = 0 and m = 1. Existence of two traveling modes in the wave-guide means that work frequency is

$$f_{r1} < f < f_{r2} \qquad (6)$$

So, there can be two traveling incident modes in the section 1 – with amplitudes A_{0i} and A_{1i}. Reflected waves can have two traveling modes as well – with the amplitudes A_{or} and A_{1r} and they go from right to left in the section 1. So, expression for the wave field (the velocity potential Φ) in the section 1 is

$$\Phi_1(r,z) = A_{0i}e^{-ikz} + A_{0r}e^{ikz} + (A_{1i}e^{-i\beta_1 z} + A_{1r}e^{i\beta_1 z})J_0(\alpha_1 r) + \sum_{m=2}^{\infty} A_m e^{\beta_m z} J_0(\alpha_m r) \qquad (7)$$

where: k = ω/C is the wave number in unbounded acoustic medium, ω = 2πf is the angular frequency.

$$\alpha_m = \frac{\gamma_m}{R_1} \quad m = 1, 2, ... \quad \beta_0 = k; \;\; \beta_1 = \sqrt{k^2 - \alpha_1^2}; \;\; \beta_2 = \sqrt{\alpha_2^2 - k^2}; \quad ... \qquad (8)$$

The index m can be considered as the modal index. Wave guide modes can be traveling and not traveling (local). If the wave number β is real – the corresponding mode is a traveling mode. If β is imaginary – the corresponding mode is a component of local field and it is not a traveling mode. All terms with the index m > 1 in the expression (7) represent local (not traveling) waves at the discontinuity. If frequency is less than f_{r1}, there can exist only one traveling mode with the modal index m = 0 (there can exist one incident and one reflected wave in the section 1).

All above regarding expression (7) can be said in other words. If work frequency meets with expression (6) there can be two traveling modes – they are represented by terms with amplitudes A_0 and A_1. Local wave field at the discontinuity (near field) is described by all terms with amplitudes A_m (m > 1). Expression (7) can be used to build wave field of a discontinuity or wave field of a source in the acoustic wave-guide.

Velocity (V) of the medium can be obtained from expression (7) in very traditional way

$$V = -grad(\Phi) \quad \rightarrow \quad V_Z^{(1)} = -\frac{\partial \Phi_1}{\partial z} \tag{9}$$

Wave field in the section 2 (z > 0) contains transmitted traveling mode(s) plus local (not traveling) waves at the discontinuity (near field of the discontinuity in the section 2). Traveling modes go from left to right – along the z-axis. If radius R_2 of the wave-guide in the section 2 is big enough for two traveling modes (meets to expression (6))

$$\Phi_2(r,z) = B_0 e^{-ikz} + B_1 e^{-i\delta_1 z} J_0(\varepsilon_1 r) + \sum_{n=2}^{\infty} B_n e^{-\delta_n z} J_0(\varepsilon_n r) \tag{10}$$

where ε_n and δ_n are radial and axial wave numbers, correspondently – in analogy with expressions (8).

The boundary conditions at the discontinuity (at z = 0) have mixed form

- axial component of the medium velocity (V_Z) should be zero over the rigid wall:
- continuity of the potential (Φ) and the axial medium velocity (V_z) in the pass at the junction.

First part of the above boundary conditions is very simple and may be realized, for example, with the routine collocation method. The second part – continuity – is not so simple. Possible mathematical formulation of the continuity for such problems has been applied in [1]. Let us consider it briefly for the case 'transmission into smaller tube'. The Bessel's functions $J_0(\varepsilon_n r)$ are the orthogonal system at $0 < r < R_2$. Hence, all coefficients B_n can be obtained directly with the Fourier-Bessel transform of the wave field in the section 1 at z = 0. This routine mathematical procedure provides continuity "automatically" – but there is one very important point. Continuity of the potential provides one set of coefficients B_n^f for the wave field in the section 2, and continuity of axial velocity – another set B_n^V. We can not expect identity of all corresponding elements in the both sets. But identity should be. Hence, it needs to impose such identity

$$B_n^f - B_n^V = 0 \qquad n = 0, 1, 2, \ldots \tag{11}$$

Exp. (11) is possible formulation of the continuity for the potential and the medium velocity in the pass. This formulation of the continuity is applied below for each problem.

To proceed further with analysis of discontinuities, we need a good model of small monopole source of velocity in the wave-guide. Appropriate model was already applied in [1] – it is small piston-type monopole source of velocity with small radius

R_S. R_S should be much less than the radius of the wave-guide and much less than one wave-length (λ) as well – in other way we should consider interaction between reflected waves and the source. Later we will see one more reason to consider very small source – at condition $R_S \to 0$ - it is a model of the point-type source for acoustic wave-guide. It is possible to employ here traditional point source with spherical symmetry, but suggested piston-type model is much more convenient for calculations in wave-guide problems. These models are equivalent at $R_S \to 0$. In this work we consider waves with axial symmetry, hence, the source must be located in such manner that allows to keep axial symmetry in the structure – at r = 0.

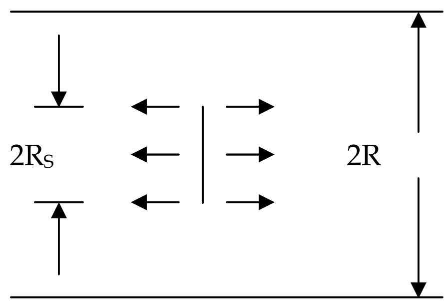

Figure 2. Model of piston-type monopole source of velocity in acoustic wave-guide with radius R.

Traditionally, small monopole source of velocity is described by the parameter Q – 'source strength'. We follow this tradition: velocity of acoustic medium over the surface of the piston's source is $Q/(\pi R_S^2)$. It is normal to the piston's surface component of the medium velocity (axial component). This source generates outgoing waves in the wave-guide and this wave field is an even function of z. Actually, such source is a double piston: one piston radiates wave in positive direction of axis z, another one – in negative direction. We will consider only one half of such wave-field, because another half can not interact with the discontinuity at z = 0 (hence, we do not introduce such components in the expression for Φ_1). So, parameter Q is the source strength for right half of the considering source.

Wave field of the considering source can be built with above expression (7). We define medium velocity in the plane of velocity source. Amplitudes of generated traveling modes can be obtained with the Hankel transform: the source provides axial medium velocity $Q/(\pi R_S^2)$ at $0 < r < R_S$ and zero axial velocity at $R_S < r < R$.

$$\frac{Q}{\pi R_S^2}\int_0^{R_S} J_0(\alpha_m r)\, r\, dr = i\,\beta_m A_m \int_0^{R} J_0^2(\alpha_m r)\, r\, dr \qquad (12)$$

Expression (12) provides following important formulas for amplitudes of outgoing modes, generated by the source in wave-guide with radius R and for $R_S \to 0$ ($R_S << R$)

$$A_{0S} = \frac{Q}{i\,k\,\pi R^2}\,; \quad A_{1S} = \frac{6.165\,Q}{i\,\beta_1\,\pi R^2}\,; \quad A_{2S} = \frac{16.04\,Q}{i\,\beta_2\,\pi R^2} \quad \ldots \qquad (13)$$

Expressions of this type can be obtained for point source with axial symmetry as well, but it would be more difficult (mathematical difficulties).

It needs to note that 'rigid diaphragm with coaxial opening' may be compared with one interesting article from the Lamb's book [6] – normal incidence of plane acoustic waves on a rigid screen with periodic slots. Actually, Lamb considered related simplified plane problem in the X-Y coordinate system. But his approximate formula for reflection coefficient R – see expression below – can be considered as some estimate for discontinuity "rigid diaphragm with coaxial opening" and it demonstrates not poor accuracy [1] almost up to f_{r1}

$$R = \frac{ik\Delta}{1+ik\Delta} \qquad \Delta = \frac{2a}{\pi}\ln\left[1/\sin(\frac{\pi b}{2a})\right]$$

So, we have already necessary basis to build wave fields in both sections of the structures 'diameter step' and 'diaphragm with coaxial opening' and we can proceed further with numerical calculations – see sections 2 – 4 below.

Let us consider – briefly – the reciprocity in the acoustic. First statements about the reciprocity belong to Helmholtz (1860, in acoustics) and Maxwell (1864, in elastic frames) [7]. Rayleigh and Lamb changed Helmholtz' statement (in 1873) to form [7, 8]: "if in a space filled with air which is partly bounded by finitely extended fixed bodies and partly unbounded, sound waves may be excited at any point A; the resulting velocity potential at a second point B is the same, both in magnitude and phase, as it have been at A, had B been the source of sound". Maxwell' version was developed to the Betti' theorem (in 1872). The Rayleigh-Lamb version of the reciprocity theorem was paraphrased later many times, but all such versions did not introduce any new details. For example, Morse & Ingard [2], p.312, wrote such shorted version: "the pressure at measurement point r, caused by source at r_o, is equal to the pressure that would be measured at r_o if the source is placed at r".

There should be noted one important detail. Rayleigh-Lamb version of the reciprocity theorem does not define directly type of the source. But in all attempts to prove this version of the reciprocity theorem researchers applied such parameter as the 'source strength' – see, for example, [7] p. 198 and [2] p. 310. Parameter 'source strength' had been defined for point monopole source of velocity, hence, the source in the Rayleigh-Lamb version of the reciprocity theorem is the source of velocity.

It can be noted that existing versions of prove cover some cases in acoustics, but not all – it will be considered in the next work. There are known violations of the Rayleigh-Lamb version of the reciprocity theorem – as well. Regarding violations – see, for example, "Eisner's reciprocity paradox" in Geophysics, 48 (1983), p.1132 - 1134. But these cases of violation should not be necessary examples of reciprocity violations, they are violations just for the Rayleigh-Lamb version… General accurate analysis of the reciprocity in acoustics – it is interesting and very important problem for future. Current work is concentrated only on discontinuities in acoustic wave-guide and their reciprocity properties.

We can note: the Rayleigh-Lamb' version of the reciprocity theorem is an example of great achievement in the acoustics at the end of 19 century. But the acoustics had some development in last 130 years and now it is not clear how we can apply this old version of the reciprocity theorem to some modern problems, for example, in multi-mode cases and in cases with different types of waves. The Rayleigh-Lamb version does not contain appropriate terminology like modes and types of the acoustic waves. **The old version of the reciprocity theorem is not able to consider any relations between different modes and different types of waves** – if such relations exist…

Any local discontinuity in the acoustic wave-guide has far field with traveling modes and near (local) field. The source has its own far and near fields as well. If the discontinuity and the source are located inside their near fields – we just do not know what should be done to see reciprocity properties and how it can be done properly. Actually, this last case of near field is very not simple and it has not been solved here. But if we will find new reciprocity relations in the far field – we must underline validity of such relations in the far field and existence of still open question about reciprocity relations in the near field.

One more point: if it needs to use old version of the reciprocity theorem in the multi-mode case, it needs to take into account all modes, generated by the source, and summarize all incoming modes at the observation point. We should not forget that modes have not equal velocities in the acoustic wave-guide. Efficiency of the source in the wave-guide is getting be different for different modes. The source begins to interact with the structure and amplitudes of radiated modes depend on the geometry of the structure – see (13) above.

There is one interesting detail in the history of the reciprocity theorem. We can not find any note about the reciprocity theorem in some very serious books about acoustics. One such examples is fundamental book of Skudrzyk [3]. There are several references on the reciprocity theorem in another fundamental book – written by Morse and Ingard [2] – copyright 1968. Twenty years later (in 1988) Ingard (himself) wrote another book – Fundamentals of waves and oscillations – and there is no one reference on the reciprocity theorem at all. Probably, some researchers looked at the Rayleigh-Lamb version of the reciprocity theorem as some kind of incomplete problem in acoustics. Probably, they are right.

Chapter 7.

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